

HOMEWORK FOR WEEK OF FEB 1

All section numbers refer to Ablowitz-Fokas.

- (1) Problems 1, 2, 6, 7 in Section 2.6
- (2) In this exercise, you will evaluate the definite integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$$

using contour integration and Cauchy's integral formula.

- (a) Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx.$$

- (b) For each $R > 10$, let C_R be the following semicircular contour:

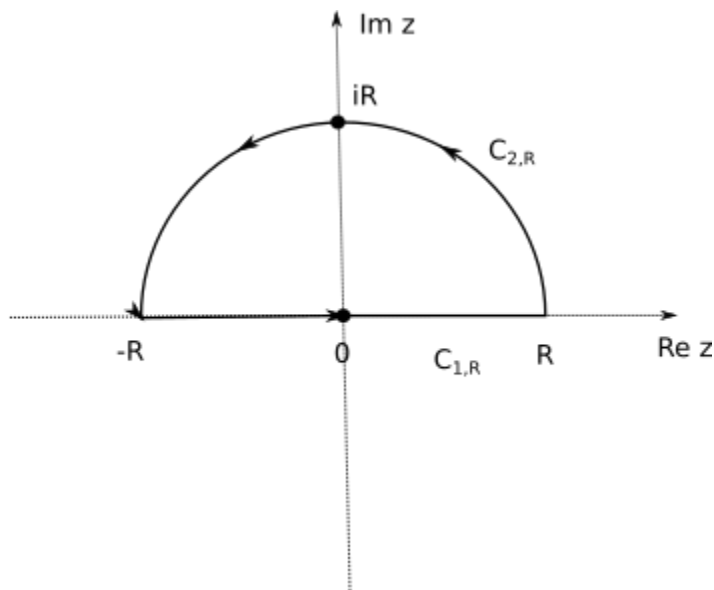


FIGURE 1. The contour $C_R = C_{1,R} + C_{2,R}$

Here, $C_{1,R} = [-R, R]$ traversed from left to right, and $C_{2,R}$ is the upper semicircle with radius R centered at 0.

Evaluate

$$\int_{C_R} \frac{e^{iz}}{1+z^2} dz.$$

(c) Note that

$$\int_{C_R} \frac{e^{iz}}{1+z^2} dz = \int_{C_{1,R}} \frac{e^{iz}}{1+z^2} dz + \int_{C_{2,R}} \frac{e^{iz}}{1+z^2} dz.$$

Using an ML bound, evaluate the limit

$$\lim_{R \rightarrow \infty} \int_{C_{2,R}} \frac{e^{iz}}{1+z^2} dz.$$

It will help here to recall the triangle inequality

$$||z| - |w|| \leq |z + w| \leq |z| + |w|, \quad z, w \in \mathbb{C}.$$

(d) Take the limit as $R \rightarrow \infty$ on both sides of the equation in part (c), and deduce the value of

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx.$$